Time-dependent gap Hele-Shaw cell with a ferrofluid: Evidence for an interfacial singularity inhibition by a magnetic field

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We consider the flow of a ferrofluid droplet in a Hele-Shaw cell with a time-dependent gap width. When the surface tension and applied magnetic field are zero, interfacial instabilities develop and the droplet breaks. We execute a mode-coupling approach to the problem and focus on understanding how the development of singularities is affected by the action of an external field. Our analytical results indicate that the introduction of an azimuthal magnetic field profoundly modifies pattern formation, allowing the inhibition of interfacial singularities. We suggest the magnetic field can be used as a controllable parameter to discipline singular behavior.

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I. INTRODUCTION

The development of finite-time singularities is of fundamental importance to a broad class of hydrodynamic problems, such as the ones related to distributions of vorticity evolving under Euler's equation [1], jet breakup [2], and droplet fission/snap-off [3]. Within this group of problems, the dynamics of the interface between viscous fluids confined in a Hele-Shaw cell (Saffman-Taylor problem) has received much attention [4–7]. In the absence of surface tension, these constrained flows are known to form cusp singularities and droplet fission at the fluid-fluid interface. Recently, an interesting work by Magdaleno *et al.* [6] studied the possibility of preventing cusp singularies for zero surface tension flows in a rotating Hele-Shaw cell. They have shown that for a subclass of exact solutions there is a critical rotation rate above which cusp formation is suppressed. Interestingly, it has been found in Ref. [6] that such a critical value for the rotation rate can be predicted by linear stability calculations. These results open up the possibility of the existence of similar types of control parameters which could inhibit the formation of finite-time singularies in other important confined flow systems.

A couple of years ago, Shelley and collaborators [7] studied another variant of the traditional Saffman-Taylor problem, and examined the dynamical evolution of a fluid drop in a Hele-Shaw cell with a time-dependent gap width. In such a cell the pressure gradient within the fluid is due to the lifting of the upper plate, leading to the formation of visually striking fingering patterns. The sophisticated numerical simulations performed in Ref. [7] revealed that, in the absence of surface tension, a dumbbell-shaped droplet would fission into two, characterizing a fissioning instability. The fluid flow in lifting Hele-Shaw cells is not only intrinsically interesting, but also of considerable importance to adhesion related problems [8–11]. Due to the practical and academic relevance of the lifting cell problem it is of interest to study ways of controlling emerging interfacial singularies.

In this work we study the evolution of a fluid droplet in a time-dependent gap Hele-Shaw cell, and consider the case in which the fluid used is a *ferrofluid* [12,13]. Ferrofluids are colloidal suspensions of nanometer-sized magnetic particles suspended in a nonmagnetic carrier fluid. These fluids are typically Newtonian and behave superparamagnetically. We investigate the situation in which the ferrofluid droplet evolves under the influence of a simple magnetic field configuration exhibiting azimuthal symmetry, produced by a current-carrying wire perpendicular to the cell plates. We perform a weakly nonlinear analysis of the problem, and find theoretical evidence indicating that the azimuthal magnetic field could be used to inhibit the emergence of interfacial singularities, even when surface tension is zero. One must exercise caution in using a weakly nonlinear approach to deal with the zero surface tension case, which presents subtle singular effects [14]. On the other hand, the present weakly nonlinear analysis serves as an alternative analytical tool to tackle the problem, being nonperturbative in surface tension. Remarkably, it has been shown recently that weakly nonlinear predictions of the Saffman-Taylor problem at low orders are compared satisfactorily to exact solutions for zero and nonzero surface tension cases [15]. The magnetically monitored process we present here certainly add a welcome versatility to usual singularity formation problems in nonmagnetic fluids, allowing the emergence of a systematic way of controlling singular behavior using ferrofluids and appropriate magnetic fields.

The layout of the rest of the paper is as follows: Section II formulates our theoretical approach. We perform a Fourier decomposition of the interface shape, and from a modified form of Darcy's law study the influence of an azimuthal magnetic field on the development of interfacial patterns in a time-dependent gap Hele-Shaw cell. Coupled, nonlinear, ordinary differential equations governing the time evolution of Fourier amplitudes are derived in both nonzero and zero surface tension cases. Section III discusses linear and weakly nonlinear dynamics, focusing on the zero surface tension limit. Section III A briefly discusses our linear stability re-

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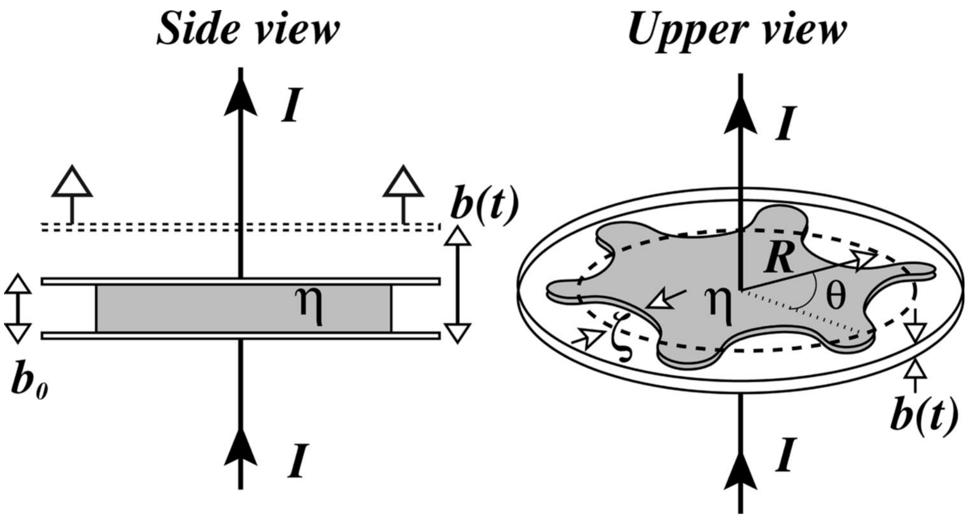

FIG. 1. Schematic representation of a time-dependent gap Hele-Shaw cell with a ferrofluid. The azimuthal magnetic field is produced by a long, straight wire carrying an electric current I.

sults, which suggest control of interfacial singularities by magnetic means. In Sec. III B we show that some important interfacial features can indeed be predicted and more quantitatively explained by our analytical mode-coupling approach. At second order we describe a finger competition phenomenon, and use it to propose a mechanism responsible for the inhibition of interfacial singularities by an azimuthal magnetic field. Our conclusions are summarized in Sec. IV.

II. THE MODE-COUPLING EQUATION

Figure 1 sketches the geometry of the lifting cell problem. Consider an incompressible ferrofluid of viscosity η located between two narrowly spaced flat plates. The outer fluid is nonmagnetic, and of negligible viscosity. The initial plate spacing is represented by b_0, and at a given time t the plate-plate distance is denoted by $b=b(t)$. A long, straight current-carrying wire is directed along the axis perpendicular to the plates. The magnetic field produced is $\mathbf{H}=I/(2\pi r)\hat{\mathbf{e}}_\theta$, where r is the distance from the wire, I represents the electric current, and $\hat{\mathbf{e}}_\theta$ is a unit vector in the azimuthal direction.

To investigate the dynamical evolution of the interface in a time-dependent gap Hele-Shaw cell, we describe its perturbed shape as $\mathcal{R}(\theta,t)=R(t)+\zeta(\theta,t)$, where $\zeta(\theta,t)=\sum_{n=-\infty}^{+\infty}\zeta_n(t)\exp(in\theta)$, represents the net interface perturbation with Fourier amplitudes $\zeta_n(t)$, and discrete azimuthal wave numbers $n=0,\pm1,\pm2,\ldots$. The unperturbed ferrofluid interface has initial and final radii defined as R_0 and $R=R(t)$, respectively. We consider a current-carrying wire of negligible radius, so that the conservation of ferrofluid volume leads to the useful relation $R^2b=R_0^2b_0$, where both R and b are *time dependent*. Notice that the Fourier expansion ζ includes the $n=0$ mode, with $\zeta_0=-(1/2R)\sum_{n\neq0}|\zeta_n(t)|^2$.

For the quasi-two-dimensional geometry of the Hele-Shaw cell, we employ the lubrication approximation and reduce the three-dimensional flow to an equivalent two-dimensional one by averaging over the direction perpendicular to the plates. We assume that the ferrofluid is uniformly magnetized and that its magnetization is collinear with the external field $\mathbf{M}=\chi\mathbf{H}$ [16,17], where χ is the constant magnetic susceptibility. This amounts to neglecting the demagnetizing field relative to the applied field and can be justified for low magnetic susceptibility of the ferrofluid, or for large applied fields that saturate the ferrofluid magnetization. It can also be justified for very thin ferrofluid films when the field is parallel to the plane of the cell.

As in the traditional Hele-Shaw problem, the flow in the ferrofluid is potential, $\mathbf{v}=-\nabla\phi$, but now with a velocity potential given by a modified Darcy's law [18]

$$\phi=\frac{b^2}{12\eta}[p-\Psi], \tag{1}$$

where p is the hydrodynamic pressure in the ferrofluid, $\Psi=\mu_0\chi H^2/2$ is a scalar potential containing the magnetic contribution, and μ_0 is the free-space permeability. In addition to the inclusion of the magnetic term in Eq. (1), we still have to consider a modified incompressibility condition of the ferrofluid, to account for the lifting of the upper plate [7] $\nabla\cdot\mathbf{v}=-\dot{b}(t)/b(t)$, where the overdot denotes total time derivative. So, in contrast to the usual Darcy's law case, the velocity potential (1) is *no longer Laplacian* and satisfies a Poisson equation

$$\nabla^2\phi=\frac{\dot{b}(t)}{b(t)}, \tag{2}$$

where its right-hand side depends only on time. As a consequence of the latter, the solution of Eq. (2) differs from being harmonic by only the simple particular solution $\bar{\phi}=\dot{b}r^2/(4b)$. The problem is then specified by two boundary conditions: (i) $p|_{\mathcal{R}}=\gamma\,\kappa$, which expresses the pressure jump at the interface, where κ denotes the interface curvature, and γ is the surface tension; and (ii) the kinematic boundary condition, which states that the normal components of each fluid's velocity $\mathbf{v}_n=-\hat{\mathbf{n}}\cdot\nabla\phi$ are continuous at the interface, where $\hat{\mathbf{n}}$ is the unit normal pointing outward.

We adapt a weakly nonlinear approach originally developed to study the traditional fixed-gap Hele-Shaw problem $(\dot{b}=0)$ with nonmagnetic fluids $(M=0)$ [19], to the current time-dependent gap situation with ferrofluids. We define Fourier expansions for the velocity potentials obeying Eq.

(2), and use the boundary conditions to express ϕ in terms of ζ_n. After some lengthy algebra, we obtain the *dimensionless* mode-coupling equation for the system (for $n \neq 0$)

$$\dot{\zeta}_n = \lambda(n)\zeta_n + \sum_{n' \neq 0} [F(n,n')\zeta_{n'}\zeta_{n-n'} + G(n,n')\dot{\zeta}_{n'}\zeta_{n-n'}], \tag{3}$$

where

$$\lambda(n) = \left[\frac{1}{2}\frac{\dot{b}}{b}(|n|-1) - \frac{\sigma b^2}{R^3}|n|(n^2-1) - |n|N_B\frac{b^2}{R^4}\right] \tag{4}$$

denotes the linear growth rate, and

$$F(n,n') = \frac{1}{R}\left\{\frac{1}{2}\frac{\dot{b}}{b}\left[|n|\left(\text{sgn}(nn') - \frac{1}{2}\right) - 1\right] - \frac{\sigma b^2}{R^3}|n|\left[1 - \frac{n'}{2}(3n'+n)\right] + \frac{3}{2}|n|N_B\frac{b^2}{R^4}\right\}, \tag{5}$$

$$G(n,n') = \frac{1}{R}\{|n|[\text{sgn}(nn') - 1] - 1\} \tag{6}$$

represent second-order mode-coupling terms. The sgn function equals ± 1 according to the sign of its argument. In Eq. (3) in-plane lengths, $b(t)$, and time are rescaled by $L_0 = 2R_0$, b_0, and the characteristic time $T = b_0/|\dot{b}(0)|$, respectively. The parameter $\sigma = \gamma b_0^3/[12\eta|\dot{b}(0)|L_0^3]$ denotes the dimensionless surface tension, and $N_B = \mu_0 \chi I^2 b_0^3/[48\pi^2\eta|\dot{b}(0)|L_0^4]$ represents the dimensionless magnetic Bond number. From now on, we work with the dimensionless version of the equations. We use Eq. (3) to examine how the scenario of finite-time singularities could be modified by the presence of an external magnetic field.

III. DISCUSSION

A. First order (linear stage)

Although at the level of purely linear analysis we do not expect to fully explain or understand the development of cusp singularities and droplet fissioning, some useful information may still be extracted from the linear growth rate (4). Hereinafter we assume that $\sigma = 0$ and consider a destabilizing driving $\dot{b}(t) > 0$. As in Ref. [7] we assume an exponentially increasing gap width $b(t) = \exp t$. This is precisely the ideal plate separation profile used in related adhesion probe-tack tests [10], since it provides a more uniform kinematics and nearly constant strain rate.

By inspecting Eq. (4) we notice that, in the absence of the magnetic field ($N_B = 0$) we have the traditional ill-posedness associated to an unregularized Saffman-Taylor instability. However, if $N_B \neq 0$ we observe that the magnetic term is always stabilizing. As time progresses the magnetic term increases $[\sim b^4(t)]$ and eventually stabilizes the system. Note that the azimuthal symmetry and radial gradient of the magnetic field will result in a magnetic force directed radially inward [18]. This force tends to stabilize the fingering instabilities arising at the ferrofluid interface, as the outer fluid enters into the system during the lifting of the upper plate. This peculiar magnetically induced stabilizing mechanism suggests that it is conceivable to have a nontrivial evolution starting from an unstable interface, but not necessarily developing finite-time singularities.

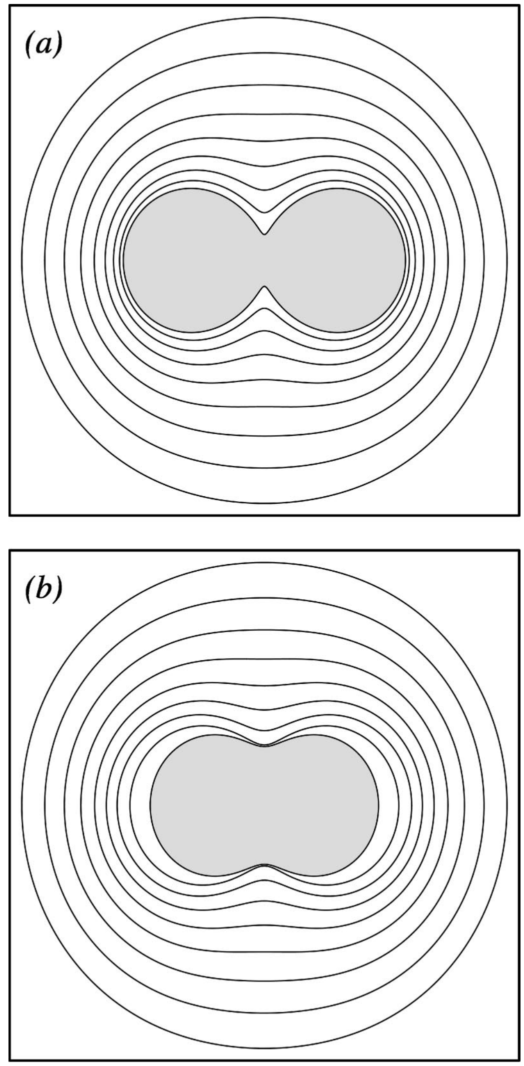

FIG. 2. Linear evolution of the interface using Eq. (3) for $n = 2$ and $0 \leq t \leq 2$ in intervals of 0.25 when (a) $N_B = 0$ and (b) $N_B = 2.5 \times 10^{-5}$.

To illustrate the overall effect of the magnetic field on the formation of finite-time singularities, we show in Fig. 2 time overlaid plots of the linear interface evolution, obtained by integrating the first term on the right-hand side of Eq. (3), for $n = 2$, and $0 \leq t \leq 2$, with equally spaced time steps of 0.25. We evolve from the initial radius $R_0 = 0.5$ with $|\zeta_n(0)| = R_0/10$. For clarity, the final droplet shape has been shaded. Figure 2(a) depicts the interface evolution in the absence of the magnetic field ($N_B = 0$). The initial circular interface evolves to a dumbbell-like shape, and tends to fission into two separate circles as described by Ref. [7]. Even though we stopped showing the evolution before the complicated pinch-off process, there is a clear evidence that a fissioning singularity tends to occur when $N_B = 0$. Note that the interface deformation grows sufficiently large that *quantitative* accuracy of any perturbative approach is doubtful. However, as discussed in detail by Gingras and Rácz [20] the linear theory is still valid as long as the pattern interfaces do not overlap. In plotting Figures 2(a) and 2(b) we have respected such validity criterion.

Figure 2(b) depicts the interface evolution for the same set of parameters used to plot Fig. 2(a), but now considering the presence of a magnetic field with $N_B=2.5\times 10^{-5}$. It is evident that the magnetic field changes considerably the ultimate motion of the interface. We recall that the magnetic terms in Eq. (3) grow exponentially with time, mimicking the intrinsic tendency towards circularization exhibited in the usual time-dependent gap Hele-Shaw flows with nonzero surface tension [7]. The most noteworthy feature in Fig. 2(b) is the absence of an imminent fission at the central droplet region. This reinforces the possibility of inhibiting fissioning instability formation with the external azimuthal magnetic field.

B. Second order (weakly nonlinear stage)

To further investigate the suggestive possibility of inhibiting singularity formation by magnetic means, we turn our attention to the weakly nonlinear terms in the mode-coupling equation (3). The numerical simulations performed in Ref. [7] for $\sigma=0$ indicate that as the interface propagates inwards, the penetrating fingers compete and the interface begins to sharpen. During this process, the formation of interfacial cusps are expected. The collision of the opposing interfaces would result in a topological singularity, producing the incipient breakup of the contracting droplet. Obviously, this competition effect is intrinsically nonlinear and could not be properly addressed by a purely linear stability analysis. To get analytical insight about this situation, we use our weakly nonlinear analysis to describe the competition process in lifting cells, and study the role played by the magnetic field in possibly avoiding the collision of the opposing interfaces.

Within our approach, finger competition is related to the influence of a fundamental mode n, assuming n is even, on the growth of its subharmonic mode $n/2$ [19]. As we have pointed out at the beginning of this work, it has been shown [15,19] that weakly nonlinear predictions of the Saffman-Taylor problem at second-order show good agreement with exact solutions for both zero and nonzero surface tension cases. Moreover, it has also been found that this agreement is obtained even when the weakly nonlinear evolution is described by the coupling of a small number of Fourier modes [15,19]. The inclusion of additional modes would certainly result in a more accurate description of the interface shape, but the basic growth mechanisms of the viscous fingering process (spreading, splitting, and competition) can be fairly well reproduced by using only a couple of relevant Fourier modes. For the purposes of the finger competition mechanism we propose in this work, the relevant modes are precisely n and $n/2$.

To simplify our discussion it is convenient to rewrite the net perturbation ζ in terms of cosine $[a_n=\zeta_n+\zeta_{-n}]$ and sine $[b_n=i(\zeta_n-\zeta_{-n})]$ modes. Without loss of generality we may

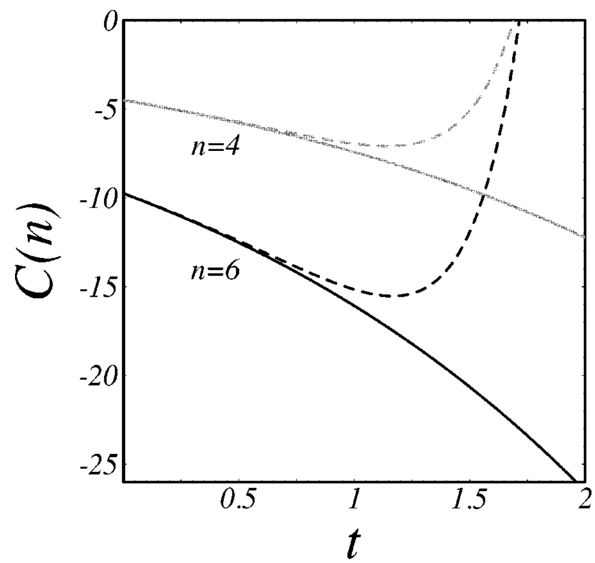

FIG. 3. $C(n)$ as a function of time for modes $n=6$ (black curves) and $n=4$ (gray curves). The magnetic Bond number is $N_B=0$ ($N_B=2.5\times 10^{-5}$) for the solid (dashed) curves.

choose the phase of the fundamental mode so that $a_n>0$ and $b_n=0$. From Eq. (3) we obtain the equations of motion for the subharmonic mode

$$\dot{a}_{n/2}=\{\lambda(n/2)+C(n)a_n\}a_{n/2}, \qquad (7)$$

$$\dot{b}_{n/2}=\{\lambda(n/2)-C(n)a_n\}b_{n/2}, \qquad (8)$$

where the function

$$C(n)=\frac{1}{2}\left[F\left(-\frac{n}{2},\frac{n}{2}\right)+\lambda(n/2)G\left(\frac{n}{2},-\frac{n}{2}\right)\right] \qquad (9)$$

disciplines finger competition.

In Fig. 3 we plot $C(n)$ as a function of time for two values of n. The solid (dashed) curves describe the behavior of $C(n)$ in the absence (presence) of the magnetic field. It is clear from Fig. 3 that $C(n)\leqslant 0$. From Eqs. (7) and (8) we verify that a negative $C(n)$ increases the growth of the sine subharmonic $b_{n/2}$ while inhibiting growth of its cosine subharmonic $a_{n/2}$. The result is an increased variability among the lengths of fingers of the outer fluid penetrating into the ferrofluid. This effect describes the competition of inward fingers.

When the magnetic field is absent (solid curves in Fig. 3), $C(n)$ is a monotonically decreasing function of time, favoring an ever increasing competition among the inward fingers, that eventually would collide resulting in a topological instability, in agreement with the numerical predictions of Ref. [7]. A completely different scenario is observed when the magnetic field is nonzero (dashed curves): initially $C(n)$ decreases with increasing t, reaches a minimum value, and subsequently increases as time advances. Eventually, $C(n)$ vanishes, meaning that the competition ceases due to the action of the magnetic field. We have verified this behavior for all values of $n\geqslant 4$. Our second-order findings suggest that the azimuthal magnetic field acts to reduce the competition among inward fingers, ultimately preventing the occurrence of interfacial singularities. These nonlinear observations are consistent with our first-order predictions (Sec. III A), regarding the stabilizing role of the applied magnetic field. Now, in addition to disciplining regular interfacial perturbations, the magnetic field seems to be able to inhibit the formation of singularities.

IV. CONCLUDING REMARKS

By employing a mode-coupling approach, we have found analytic evidence that the introduction of a ferrofluid into a lifting Hele-Shaw cell, subjected to an azimuthal applied field, may provide a magnetically induced way to inhibiting the formation of cusp and fissioning singularities in zero surface tension flows. This field-regulated behavior is predicted by our linear stability analysis, and reinforced by our weakly nonlinear results.

We point out that the controlling mechanism we suggest, and the specific predictions of our theoretical work, have not yet been checked experimentally. Considering the fundamental importance of singularity formation to many problems in fluid dynamics, we believe it would be of interest to experimentalists to study the role of magnetic fields in disciplining singular behavior in ferrofluids. An interesting possibility in this direction would involve the development of experiments in the time-dependent gap Hele-Shaw cell using *phase-separated* ferrofluids [21–23], which are magnetic liquids consisting of a phase rich in magnetic particles in suspension in another phase poor in such particles. For these magnetic fluids, it is known that near the critical point the surface tension between the two coexisting phases can be very small, tending precisely to zero at the critical point. Another possibility would be performing lifting Hele-Shaw cell experiments using *miscible* magnetic and nonmagnetic fluids [24,25].

On the theoretical side, a quantitative test of our chief results to fully nonlinear stages of interface evolution would require the calculation of exact solutions, or the elaboration of sophisticated numerical simulations capable of describing non-Laplacian flows [7,26–28]. Of course, these theoretical approaches should be appropriately adapted to characterize accurately the behavior of a ferrofluid droplet under applied magnetic field, in the zero surface tension limit. In conclusion, we hope the present work will impel further (experimental and theoretical) studies on this fruitful research topic, which would allow the check of the predictions made by our linear and weakly nonlinear analyses.

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